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Alternative Approach to the Concept of Shape Invariance in Quantum Mechanics

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It is shown that in the case of shape invariance, with the use of the mixing function formalism, exact solvability of the Schrödinger equation can be obtained without recourse to the technique of supersymmetrization. Some consequences of the method are outlined and illustrated by specific examples.

1. INTRODUCTION

Shape invariance is a quite useful tool to investigate the exact solvability of the Schrödinger equation

$$H\psi_n = E_n\psi_n, \qquad H = \frac{d^2}{dx^2} - V(l, x) \tag{1}$$

in which the set of eigenfunctions $\{\psi_n\}$ and eigenvalues $\{E_n\}$ can be determined by algebraic means and when the potential $V(l, x)$ can be written as

$$V(l, x) = u^2(l, x) - u'(l, x) \tag{2}$$

where l is a parameter.

Since its discovery (Gedenshtein, 1983) it has been commonly thought that exact solvability using shape invariance can be derived only through the use of the technique of supersymmetrization.

This paper will present a different approach by showing that the same result can also be obtained from simple algebra, independent of $SU(2)$, and this may have useful consequences from the pedagogical point of view.

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2. FORMULATION

The theory of “mixing functions” has been described in previous papers (Cao, 1994, 1997), so that only a brief review of some essential features which will be needed below will be outlined here.

Consider the matrix equation

$$\phi' + F\phi = 0; \qquad F = \begin{pmatrix} u_1 & d_1 \\ 0 & u_2 \end{pmatrix}; \qquad \phi = (\phi_1, \phi_2)^+ \tag{3}$$

with the notation $\phi' = d\phi/dx$.

The u_1, u_2, and d_1 may be any analytic functions of x. The “mixing function” X is defined as

$$\phi_1 = -X\phi_2 \tag{4}$$

and the compatibility condition requires that u_1, u_2, d_1 be linked by the relation

$$d_1 = -X' + (u_2 - u_1)X \tag{5}$$

If X is the solution of the equation

$$X'' - 2u_2X' + [(u_2^2 - u_2') - (u_2^2 - u_1')]X = EX \tag{6}$$

then by differentiation of (3) and successive use of (4) and (5) we obtain the following results:

$$\phi_1'' - (u_1^2 - u_1')\phi_1 = E\phi_1, \qquad \phi_1 \simeq -X\exp\left(-\int u_2\,dx\right) \tag{7}$$

$$\phi_2'' - (u_2^2 - u_2')\phi_2 = 0$$

The problem of exact solvability of the potential $V(l, x) = u_1^2(l, x) - u_1'(l, x)$ for the case of shape invariance is thus reduced to the search for exact solutions of equation (6), where E represents the set of eigenvalues $\{E_n\}$.

3. THE ANSATZ

This can be written as

$$u^2(l, x) \pm u'(l, x) = u^2(l_1, x) \mp u'(l_1, x) + h^{\pm}(l_1) \tag{8}$$

in which $h^{\pm}(l_1)$ is a constant depending on the parameter l_1, which is related to the initial one l by a transformation T_1, i.e., $l_1 = T_1 l$. The still open question now is what should be T_n such that the above equation is exactly solvable? For the moment, we note the following cases: (a) translation $l_1 = l - 1$,

(b) scaling $l_1 = q^l$, and (c) nonscaling $l_1 = ql^p$, where q, p are parameters (Spiridonov, 1992; Barclay *et al.*, 1993).

Although the approach presented here remains valid for these cases, in this work we choose to develop the discussion within the frame of case (a), since most of the potentials of practical use in quantum mechanics can in fact be constructed from it (see, for instance, Cao, 1991, and references therein).

Consider now equation (3) in which ($l_1 = T_1 l = l - 1$)

$$u_1 = u(l, x); \qquad u_2 = u(l - n, x)$$

Here n corresponds to the nth excited stateof the potential $V(l, x) = u_1^2 - u_1'$. Therefore

$$\phi_{l,n} \simeq X_n \exp\left[-\int u(l - n, x)\, dx\right]$$

with the condition

$$X_n'' - 2u(l - n, x)X_n' + [(u^2(l - n, x) - u'(l - n, x)) - (u^2(l, x) - u'(l, x))]X_n = E_n X_n \tag{9}$$

In the case of translation, it can be verified from the above ansatz that the exact solvability problem is now reduced to the following results:

(a) The mixing function must be solution of the equation

$$X_n'' - 2u(l - n, x)X_n' + 2 \sum_{s=0}^{n-1} u'(l - s)X_n = 0 \tag{10}$$

(b) We have

$$E_n = \sum_{r=1}^{n} h(l - r)$$

4. THE ASSOCIATED PROBLEM

If in (3) we set $u_1 = -u(l, x)$ and $u_2 = u(l - n, x)$, then another system $\bar{\phi} = (\bar{\phi}_1, \phi_2)^+$ can be defined with the corresponding mixing function Y

$$\bar{\phi}_1 = -Y\phi_2$$

Proceeding exactly as above and taking into account the change of sign in (9), we obtain a second Schrödinger equation

$$\bar{\phi}_1'' - (u_1^2 + u_1')\bar{\phi}_1 = \bar{E}\phi_1; \qquad \bar{\phi}_1 \simeq -Y \exp\left[-\int u(l - n, x)\, dz\right] \tag{11}$$

The mixing function Y must satisfy the equation

$$Y''_m - 2u(l-m, x)Y'_m + 2\sum_{s=1}^{m-1} u'(l-s, x)Y_m = 0 \tag{12}$$

while

$$\overline{E}_m = \sum_{r=1}^{m} h(l-r)$$

(a) If $u_1 = u_2$, then the eigenfunction corresponding to the state $|n\rangle$ must be

$$\phi_{l,n} = -\overline{X}_n \exp\left[-\int u(l, x)\, dx\right] \tag{13}$$

in which the function $\overline{X}_n$ is a solution of the equation

$$\overline{X}''_n - 2u(l, x)\overline{X}'_n - \overline{E}_n\overline{X}_n = 0 \tag{14}$$

This aspect has already been discussed in Cao (1997) and noting from the uniticity of the eigenfunction

$$\phi_{l,n} = -\overline{X}_n \exp\left[-\int u(l, x)\, dx\right] = -X_n \exp\left[-\int u(l-n, x)\, dx\right]$$

so that

$$X_n = \overline{X}_n \exp\left\{\int [u(l-n, x) - u(l, x)]\, dx\right\}$$

X and $\overline{X}$ are identical only for the ground state $|0\rangle$, which is assumed to be normalizable and to correspond to the eigenvalue $E_0 = 0$.

(b) Likewise we have for the associated problem

$$\overline{\phi}_{l,m} = -\overline{Y}_m \exp\left[-\int u(l, x)\, dx\right]$$

where $\overline{Y}_m$ is a solution of the equation

$$\overline{Y}''_m - 2u(l, x)\overline{Y}'_m - [2u'(l, x) + \overline{E}_m]\overline{Y}_m = 0 \tag{15}$$

On the other hand, differentiating (14), we find

$$\overline{X}'''_n - 2u(l, x)\overline{X}''_n - [2u'(l, x) + E_n]X'_n = 0 \tag{16}$$

and comparing (16) and (15), it appears that they are identical if

$$\overline{X}'_n \equiv Y_m, \qquad \overline{E}_m \equiv En$$

As for the ground state $|0\rangle$, $X_0 = \text{const}$, which means that $\overline{Y}_0$ is not defined. Therefore, the only consistent choice is to take $m = n + 1$, indicating that the two eigenspectra $\{E_n\}$ and $\{\overline{E}_n\}$ are degenerate, which is a well-known result derived from supersymmetry. This is the reason why the couple $\overline{X}_n$, Y_n can be referred to as supersymmetric partners.

The behavior of the couple X_n, Y_n is, however, different:

(a) Define two operators $c^{\pm} = d/dx \mp [u(l, x) \pm u(l - n, x]$.

Then by making use of the ansatz (8) it can be checked that the two equations (10) and (12) can also be written in terms of $c^{\pm}$ as

$$c^- c^+ Y_n = E_n Y_n, \qquad c^+ c^- X_n = E_n X_n \tag{17}$$

which means that the two components of the couple (X_n, Y_n) must be linked by the coupled system:

$$E_n^{1/2} X_n = c^+ Y_n, \qquad E_n^{1/2} Y_n = c^- X_n \tag{18}$$

(b) In order to see how the above results can be used in the theory, consider again system (3), in which we set

$$u_1 = u(l - p, x), \qquad u_2 = u(l - p - m, x), \qquad p = 0, 1, 2, \ldots \tag{19}$$

Following then exactly the same reasoning, we can write the resulting Schrödinger equation as [the notations (l) and $(l - p)$ are added in order to avoid any possible confusion]

$$\phi_m^{(l-p)''} - [u^2(l - p, x) - u'(l - p, x)]\phi_m^{(l-p)} = E_m^{(l-p)}\phi_m^{(l-p)} \tag{20}$$

and corresponds to another mixing function $X_m^{(l-p)}$. Exact solvability means that this function must be solution of the following equation:

$$X_m^{(l-p)''} - 2u(l - p - m, x)X_m^{(l-p)'} + 2\sum_{s=p}^{m+p-1} u'(l - s)X_m^{(l-p)} = 0 \tag{21}$$

with eigenvalues

$$E_m^{(l-p)} = \sum_{r=p}^{m+p} h(l - r) = -E_m^{(l)} + \sum_{t=1}^{p-1} h(l - t) \tag{22}$$

As m is arbitrary, there will be no loss of generality by setting $m = n - p$ (m, n refer to the states $|m\rangle$, $|n\rangle$) and noting that $l - p - (n - p) \equiv l - n$. Therefore the above equation can be cast in the form

$$X_{n-p}^{(l-p)''} - 2u(l - n, x)X_{n-p}^{(l-p)'} + 2\sum_{s=p}^{n-1} u'(l - s, x)X_{n-p}^{(l-p)} = 0 \tag{23}$$

The special case $p = 1$ is particularly interesting since it can be verified that (23) is in fact identical to the equation (12), which itself corresponds to the associated mixing function $Y_n^{(l)}$. For this reason and up to a constant, $Y_n^{(l)} = X_{n-1}^{(l-1)}$.

Returning now to the first equation in (20), we have

$$E_n^{1/2} X_n^{(l)} = \left[\frac{d}{dx} - [u(l, x) + u(l - n, x)] \right] X_{n-1}^{(l-1)} \tag{24}$$

which provides the connection needed between the two functions $X_n^{(l)}$ and $X_{n-1}^{(l-1)}$. The analytic expression of the second one is constructed from the first one in which the quantities n and l must be replaced by $l - 1$ and $n - 1$ and the process must continue until the last step is reached with $X_0^{(l-n)} = \text{const}$.

5. THE EIGENFUNCTION

In order to check for the consistency of the present method, we analyze the case of the eigenfunctions, since by definition

$$\phi_{1,n}^{(l)} = -X_n^{(l)} \exp\left[-\int u(l - n, x)\, dx \right];$$

$$\phi_{1,n-1}^{(l-1)} = -X_{n-1}^{(l-1)} \exp\left[-\int u(l - n, x)\, dx \right]$$

Eliminating then the quantities $X_n^{(l)}$ and $X_{n-1}^{(l-1)}$ and after some simple algebra, we find

$$\phi_{1,n}^{(l)} = A^{(l)} \phi_{1,n-1}^{(l-1)} \tag{25}$$

in which $A^{(l)}$ is the operator defined by $A^{(l)} = d/dx - u(l, x)$. One may recognize in (25) the essential result obtained in supersymmetry in the case of shape invariance, $A^{(l)}$ being the ladder operator.

6. INTERPRETATION

To get a deeper insight on the method and see why it works for the case of shape invariance, it would be appropriate to return to the general case (3) in which u_1 and u_2 may be any analytic functions. Define the matrices B^-, B^+ such that

$$B^- = \begin{pmatrix} 0 & 0 \\ C^- & 0 \end{pmatrix}, \qquad B^+ = \begin{pmatrix} 0 & C^+ \\ 0 & 0 \end{pmatrix}$$

$$C^\pm = \frac{d}{dx} \mp (u_1 \pm u_2)$$

The following commutation ([,]) and anticommutation ({, }) rules hold:

$$(B^-)^2 = (B^+)^2 = 0; \qquad \{B^-, B^+\} = \bar{h} = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}$$

$$h_1 = C^- C^+; \qquad h_2 = C^+ C^- \tag{26}$$

$$[B^-, \bar{h}] = [B^+, \bar{h}] = 0$$

$$[C^-, C^+] = 2u_1', \qquad \{C^-, C^+\} = 2(h_{\frac{1}{2}} \mp u_1')$$

More explicitly,

$$h_{\frac{1}{2}} = \frac{d^2}{dx^2} - 2u_2 \frac{d}{dx} + [(u_2^2 - u_2') - (u_1^2 \mp u_1')$$

which can be compared with relation (6), etc.

As the couple B^-, B^+ is not adjoint, this construction differs from conventional supersymmetry and the two Hamiltonians h_1, h_2 are often referred to as the "gauge Hamiltonian," the role of the gauge being played by the function u_2, which itself is arbitrary, while the function u_1 serves to define the potential $V = u_1^2 - u_1'$.

Therefore, if the couple X, Y are eigenfunctions of h_1, h_2, the potentials V and $\bar{V} = u_1^2 + u_1'$ are both exactly solvable.

In other words, it can be stated that if these two approaches are equivalent, the essence of their focus differs since the object of supersymmetry is the pair of eigenfunctions ϕ_1, $\bar{\phi}_1$ (the partner eigenfunctions), while the present method deals with the pair of mixing functions X, Y.

The relative simplicity observed in the special case of shape invariance results in fact from the intervention of the ansatz (8), but in general, and despite this equivalence, these two approaches will lead to different applications. The case of supersymmetry is already well known and need not be discussed here, while some consequences of the mixing function formalism will be analyzed below and illustrated by specific examples in the Appendix.

To summarize, the new points presented in this paper are as follows:

(a) The equivalence between the mixing function formalism and conventional supersymmetry can be established by showing that the analytic determination of the set of exact eigenfunctions and eigenvalues which constitute the hallmark of $SU(2)$ for the case of shape invariance can also be obtained independently through simple algebra.

(b) The present approach, on the other hand, reveals a different facet of the problem which is not predicted in $SU(2)$ since with the pair of mixing functions X, Y combined with repeated use of $C^{\pm}$ we may have at our disposal a convenient means to construct exact solutions of differential equations of type (10) or (12) especially when these equations cannot be handled with usual techniques. The example in the first part of the Appendix supports this point of view.

(c) These solutions can in turn be used to construct a new type of exactly solvable potential $V^{(n)}(x)$, already discussed in an earlier paper (Cao, 1997) which in the present context can be written as

$$V^{(n)}(z) = u_1^2 + u_1' + 2\,\frac{\overline{X_n'}}{\overline{X_n}}\left[\frac{\overline{X_n'}}{\overline{X_n}} - 2u_1'\right] \tag{27}$$

This is illustrated by the second example in the Appendix.

(d) Finally, for the case of shape invariance, the fact that the present method can be applied for the chain of potentials $V(l - p, x)$, $p = 1, 2, \ldots,$ corresponding to the set of eigenvalues given in (22) suggests extension of the investigation to a number of other directions of research, for instance, p-multiply degenerate systems.

APPENDIX

(a) Let $u_1(l, x) = l \tanh x$, $l > n$, $V(l, x) = l^2 - l(l + 1)/\cosh^2 x$.

The exact analytic expression (up to a constant) is given below for the first values of n:

n	X_n
0	const
1	$\tanh x$
2	$-1 + (2l - 1)\tanh^2 x$
3	$\tanh x\left[(2l - 4) - \dfrac{2l - 1}{\cosh^2 x}\right] \cdots$

(28)

It can be verified that they are solutions of equation (10) which in the present case take the form

$$X_n'' - 2(l - n)\tanh x X_n' + 2n\,\frac{l - \frac{1}{2}(n - 1)}{\cosh^2 x}\,X_n = 0$$

Remark. This approach may also be applied to a number of other types of analytic functions (u_1, u_2) and will be presented in another paper.

(b) The exact eigenfunctions and eigenvalues of the first-order generation potential $V^n(l, x)$ defined in (27) are, respectively,

$$\phi_{m,n} = N_{m,n}\lambda'_{m,n} \exp\left\{-\int \left[u(l, x) - \frac{\bar{X}'_1}{\bar{X}_1}\right] dx\right\}$$
$$E_{m,n} = E_m - 2E_n \tag{29}$$

$N_{m,n}$ is the constant of normalization; m and n correspond to the excited states $|m\rangle$ and $|n\rangle$ of the "parent" potential $V = u_1^2 - u_1'$, while the quantity $\lambda_{m,n}$ is

$$\lambda_{m,n} = \frac{\bar{X}_m}{\bar{X}_n}$$

The relation between the functions X_n and $\bar{X}_n$ is

$$\bar{X}_n = X_n \exp\left\{-\int [u(l - h, x) - u(l, x)]\, dx\right\} \tag{30}$$

Therefore, making use of (30), (29), and (28), we can infer the eigenfunction $\phi_{m,n}$ corresponding to the state $|m, n\rangle$ of the potential $V^{(n)}$ defined in (27).

For instance, with $n = 1$, this potential is

$$V^{(l)}(x) = l^2 - 4l + 2 - \frac{l(l - 1)}{\cosh^2 x} + \frac{2}{\sinh^2 x}$$

The first two states $|2, 1\rangle$ and $|3, 1\rangle$ correspond to the eigenfunctions

$$\phi_{2,1} = N_{2,1} \frac{\sinh x}{\cosh^{l-1} x}\left[2(l - 1) + \frac{1}{\sinh^2 x}\right]$$
$$\phi_{3,1} = N_{3,1} \frac{\sinh^2 x}{\cosh^{l-1} x}$$

The first one is obviously nonnormalizable, while the second one always is if $l > 3$. It corresponds to the eigenvalue

$$E_{3,1} = -2l + 7$$

as can be checked by direct substitution in the Schrödinger equation

$$\phi''_{3,1} - V^{(l)}\phi_{3,1} = E_{3,1}\phi_{3,1}$$

Higher order generation potentials of type $V^{(m,n)}$ can also be constructed, but we shall not discuss this aspect here. For the moment, it may be instructive to point out the following remark:

If the parent potential $V = u_1^2 - u_1'$ is shape invariant, its first-order generation potential $V^{(1)}$ defined in (27), while exactly solvable, does not necessarily conserve the same property.

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